## Section A: Pure Mathematics

**1** The sequence of real numbers $u_1$, $u_2$, $u_3$, ... is defined by

$$u_1 = 2\,, \qquad \text{and} \qquad u_{n+1} = k - \frac{36}{u_n} \quad \text{for } n \geqslant 1, \qquad (*)$$

where $k$ is a constant.

(i) Determine the values of $k$ for which the sequence $(*)$ is:

(a) constant;

(b) periodic with period 2;

(c) periodic with period 4.

(ii) In the case $k = 37$, show that $u_n \geqslant 2$ for all $n$. Given that in this case the sequence $(*)$ converges to a limit $\ell$, find the value of $\ell$.

**2** Using the series

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\,,$$

show that $\mathrm{e} > \frac{8}{3}$.

Show that $n! > 2^n$ for $n \geqslant 4$ and hence show that $\mathrm{e} < \frac{67}{24}$.

Show that the curve with equation

$$y = 3\mathrm{e}^{2x} + 14\ln(\tfrac{4}{3} - x)\,, \qquad x < \tfrac{4}{3}$$

has a minimum turning point between $x = \frac{1}{2}$ and $x = 1$ and give a sketch to show the shape of the curve.

**3** (i) Show that $\left(5+\sqrt{24}\,\right)^4 + \dfrac{1}{\left(5+\sqrt{24}\,\right)^4}$ is an integer.

Show also that

$$0.1 < \frac{1}{5+\sqrt{24}} < \frac{2}{19} < 0.11\,.$$

Hence determine, with clear reasoning, the value of $\left(5+\sqrt{24}\right)^4$ correct to four decimal places.

(ii) If $N$ is an integer greater than 1, show that $\left(N+\sqrt{N^2-1}\right)^k$, where $k$ is a positive integer, differs from the integer nearest to it by less than $\left(2N-\frac{1}{2}\right)^{-k}$.

**4** By making the substitution $x = \pi - t$, show that

$$\int_0^{\pi} x\mathrm{f}(\sin x)\mathrm{d}x = \tfrac{1}{2}\pi\int_0^{\pi} \mathrm{f}(\sin x)\mathrm{d}x\,,$$

where $\mathrm{f}(\sin x)$ is a given function of $\sin x$.

Evaluate the following integrals:

(i) $\displaystyle\int_0^{\pi} \frac{x\sin x}{3+\sin^2 x}\,\mathrm{d}x\,;$

(ii) $\displaystyle\int_0^{2\pi} \frac{x\sin x}{3+\sin^2 x}\,\mathrm{d}x\,;$

(iii) $\displaystyle\int_0^{\pi} \frac{x\left|\sin 2x\right|}{3+\sin^2 x}\,\mathrm{d}x\,.$

**5** The notation $\lfloor x \rfloor$ denotes the greatest integer less than or equal to the real number $x$. Thus, for example, $\lfloor \pi \rfloor = 3$, $\lfloor 18 \rfloor = 18$ and $\lfloor -4.2 \rfloor = -5$.

(i) Two curves are given by $y = x^2 + 3x - 1$ and $y = x^2 + 3\lfloor x \rfloor - 1$. Sketch the curves, for $1 \leqslant x \leqslant 3$, on the same axes.

Find the area between the two curves for $1 \leqslant x \leqslant n$, where $n$ is a positive integer.

(ii) Two curves are given by $y = x^2 + 3x - 1$ and $y = \lfloor x \rfloor^2 + 3\lfloor x \rfloor - 1$. Sketch the curves, for $1 \leqslant x \leqslant 3$, on the same axes.

Show that the area between the two curves for $1 \leqslant x \leqslant n$, where $n$ is a positive integer, is

$$\tfrac{1}{6}(n-1)(3n+11)\,.$$

**6** By considering a suitable scalar product, prove that

$$(ax + by + cz)^2 \leqslant (a^2 + b^2 + c^2)(x^2 + y^2 + z^2)$$

for any real numbers $a$, $b$, $c$, $x$, $y$ and $z$. Deduce a necessary and sufficient condition on $a$, $b$, $c$, $x$, $y$ and $z$ for the following equation to hold:

$$(ax + by + cz)^2 = (a^2 + b^2 + c^2)(x^2 + y^2 + z^2)\,.$$

(i) Show that $(x + 2y + 2z)^2 \leqslant 9(x^2 + y^2 + z^2)$ for all real numbers $x$, $y$ and $z$.

(ii) Find real numbers $p$, $q$ and $r$ that satisfy both

$$p^2 + 4q^2 + 9r^2 = 729 \quad \text{and} \quad 8p + 8q + 3r = 243\,.$$

**7** An ellipse has equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Show that the equation of the tangent at the point $(a\cos\alpha, b\sin\alpha)$ is

$$y = -\frac{b\cot\alpha}{a}\,x + b\,\mathrm{cosec}\,\alpha\,.$$

The point $A$ has coordinates $(-a, -b)$, where $a$ and $b$ are positive. The point $E$ has coordinates $(-a, 0)$ and the point $P$ has coordinates $(a, kb)$, where $0 < k < 1$. The line through $E$ parallel to $AP$ meets the line $y = b$ at the point $Q$. Show that the line $PQ$ is tangent to the above ellipse at the point given by $\tan(\alpha/2) = k$.

Determine by means of sketches, or otherwise, whether this result holds also for $k = 0$ and $k = 1$.

**8** Show that the line through the points with position vectors $\mathbf{x}$ and $\mathbf{y}$ has equation

$$\mathbf{r} = (1 - \alpha)\mathbf{x} + \alpha\mathbf{y}\,,$$

where $\alpha$ is a scalar parameter.

The sides $OA$ and $CB$ of a trapezium $OABC$ are parallel, and $OA > CB$. The point $E$ on $OA$ is such that $OE : EA = 1 : 2$, and $F$ is the midpoint of $CB$. The point $D$ is the intersection of $OC$ produced and $AB$ produced; the point $G$ is the intersection of $OB$ and $EF$; and the point $H$ is the intersection of $DG$ produced and $OA$. Let $\mathbf{a}$ and $\mathbf{c}$ be the position vectors of the points $A$ and $C$, respectively, with respect to the origin $O$.

(i) Show that $B$ has position vector $\lambda\mathbf{a} + \mathbf{c}$ for some scalar parameter $\lambda$.

(ii) Find, in terms of $\mathbf{a}$, $\mathbf{c}$ and $\lambda$ only, the position vectors of $D$, $E$, $F$, $G$ and $H$. Determine the ratio $OH : HA$.

## Section B: Mechanics

**9** A painter of weight $kW$ uses a ladder to reach the guttering on the outside wall of a house. The wall is vertical and the ground is horizontal. The ladder is modelled as a uniform rod of weight $W$ and length $6a$.

The ladder is not long enough, so the painter stands the ladder on a uniform table. The table has weight $2W$ and a square top of side $\frac{1}{2}a$ with a leg of length $a$ at each corner. The foot of the ladder is at the centre of the table top and the ladder is inclined at an angle $\arctan 2$ to the horizontal. The edge of the table nearest the wall is parallel to the wall.

The coefficient of friction between the foot of the ladder and the table top is $\frac{1}{2}$. The contact between the ladder and the wall is sufficiently smooth for the effects of friction to be ignored.

(i) Show that, if the legs of the table are fixed to the ground, the ladder does not slip on the table however high the painter stands on the ladder.

(ii) It is given that $k = 9$ and that the coefficient of friction between each table leg and the ground is $\frac{1}{3}$. If the legs of the table are not fixed to the ground, so that the table can tilt or slip, determine which occurs first when the painter slowly climbs the ladder.

[Note: $\arctan 2$ is another notation for $\tan^{-1} 2$.]

**10** Three particles, $A$, $B$ and $C$, of masses $m$, $km$ and $3m$ respectively, are initially at rest lying in a straight line on a smooth horizontal surface. Then $A$ is projected towards $B$ at speed $u$. After the collision, $B$ collides with $C$. The coefficient of restitution between $A$ and $B$ is $\frac{1}{2}$ and the coefficient of restitution between $B$ and $C$ is $\frac{1}{4}$.

(i) Find the range of values of $k$ for which $A$ and $B$ collide for a second time.

(ii) Given that $k = 1$ and that $B$ and $C$ are initially a distance $d$ apart, show that the time that elapses between the two collisions of $A$ and $B$ is $\dfrac{60d}{13u}$.

**11** A projectile of unit mass is fired in a northerly direction from a point on a horizontal plain at speed $u$ and an angle $\theta$ above the horizontal. It lands at a point $A$ on the plain. In flight, the projectile experiences two forces: gravity, of magnitude $g$; and a horizontal force of constant magnitude $f$ due to a wind blowing from North to South. Derive an expression, in terms of $u$, $g$, $f$ and $\theta$ for the distance $OA$.

(i) Determine the angle $\alpha$ such that, for all $\theta > \alpha$, the wind starts to blow the projectile back towards $O$ before it lands at $A$.

(ii) An identical projectile, which experiences the same forces, is fired from $O$ in a northerly direction at speed $u$ and angle $45^\circ$ above the horizontal and lands at a point $B$ on the plain. Given that $\theta$ is chosen to maximise $OA$, show that

$$\frac{OB}{OA} = \frac{g - f}{\sqrt{g^2 + f^2} - f} \,.$$

Describe carefully the motion of the second projectile when $f = g$.

## Section C: Probability and Statistics

**12** A cricket team has only three bowlers, Arthur, Betty and Cuba, each of whom bowls 30 balls in any match. Past performance reveals that, on average, Arthur takes one wicket for every 36 balls bowled, Betty takes one wicket for every 25 balls bowled, and Cuba takes one wicket for every 41 balls bowled.

(i) In one match, the team took exactly one wicket, but the name of the bowler was not recorded. Using a binomial model, find the probability that Arthur was the bowler.

(ii) Show that the average number of wickets taken by the team in a match is approximately 3. Give with brief justification a suitable model for the number of wickets taken by the team in a match and show that the probability of the team taking at least five wickets in a given match is approximately $\frac{1}{5}$.

[You may use the approximation $e^3 = 20$.]

**13** I know that ice-creams come in $n$ different sizes, but I don't know what the sizes are. I am offered one of each in succession, in random order. I am certainly going to choose one — the bigger the better — but I am not allowed more than one. My strategy is to reject the first ice-cream I am offered and choose the first one thereafter that is bigger than the first one I was offered; if the first ice-cream offered is in fact the biggest one, then I have to put up with the last one, however small.

Let $P_n(k)$ be the probability that I choose the $k$th biggest ice-cream, where $k = 1$ is the biggest and $k = n$ is the smallest.

(i) Show that $P_4(1) = \frac{11}{24}$ and find $P_4(2)$, $P_4(3)$ and $P_4(4)$.

(ii) Find an expression for $P_n(1)$.

**14** Sketch the graph of $y = \dfrac{1}{x \ln x}$ for $x > 0$, $x \neq 1$. You may assume that $x \ln x \to 0$ as $x \to 0$.

The continuous random variable $X$ has probability density function

$$\mathrm{f}(x) = \begin{cases} \dfrac{\lambda}{x \ln x} & \text{for } a \leqslant x \leqslant b\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where $a$, $b$ and $\lambda$ are suitably chosen constants.

(i) In the case $a = 1/4$ and $b = 1/2$, find $\lambda$.

(ii) In the case $\lambda = 1$ and $a > 1$, show that $b = a^{\mathrm{e}}$.

(iii) In the case $\lambda = 1$ and $a = \mathrm{e}$, show that $\mathrm{P}(\mathrm{e}^{3/2} \leqslant X \leqslant \mathrm{e}^2) \approx \frac{31}{108}$.

(iv) In the case $\lambda = 1$ and $a = \mathrm{e}^{1/2}$, find $\mathrm{P}(\mathrm{e}^{3/2} \leqslant X \leqslant \mathrm{e}^2)$.